

IDENTITIES OF SUM OF TWO PI-ALGEBRAS IN THE CASE OF POSITIVE CHARACTERISTIC

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ABSTRACT. We consider the following question posted by K.I. Beidar and A.V. Mikhalev for a ring $R = R_1 + R_2$: is it true that if subrings R_1 and R_2 satisfy polynomial identities, then R also satisfies a polynomial identity?

Keywords: Sum of rings, algebras with polynomial identities, positive characteristic.

2010 MSC: 16R10.

1. INTRODUCTION

We assume that \mathbb{F} is an infinite field of arbitrary characteristic $p = \text{char } \mathbb{F} \geq 0$. All vector spaces, algebras and modules are over \mathbb{F} and all algebras are associative. An algebra may not have a unity.

For a commutative ring K denote by $K\langle X \rangle$ the free K -module, freely generated by all non-empty products of letters x_1, x_2, \dots . Given a ring R of the characteristic $m \geq 0$, a polynomial identity of R is an element $f(x_1, \dots, x_n)$ of $\mathbb{Z}_m\langle X \rangle$ such that $f(r_1, \dots, r_n) = 0$ in R for all $r_1, \dots, r_n \in R$, where \mathbb{Z}_m stands for $\mathbb{Z}/m\mathbb{Z}$ in case $m > 0$ and $\mathbb{Z}_0 = \mathbb{Z}$. Similarly we can define a polynomial identity of an \mathbb{F} -algebra A as an element of $\mathbb{F}\langle X \rangle$. A ring (an algebra, respectively) which satisfies a polynomial identity is called a PI-ring (PI-algebra, respectively). For short, polynomial identities are called identities.

Consider a ring R with two subrings R_1 and R_2 satisfying condition $R = R_1 + R_2$ (i.e. any element of R is equal to the sum $r_1 + r_2$ for some $r_1 \in R_1$ and $r_2 \in R_2$). In 1995 K.I. Beidar and A.V. Mikhalev posted the following question which is still open (see [2]):

Is it true that if R_1 and R_2 satisfy polynomial identities, then R also satisfies a polynomial identity?

The same question can also be asked for algebras over a base field \mathbb{F} .

The positive answer to this question is known in many cases. O.H. Kegel [4] established that if R_1 and R_2 are nilpotent, then R is also nilpotent. By the result of Bahturin and Giambruno [1], if R_1 and R_2 are commutative rings, then R satisfies the identity $[x, y][a, b] = 0$, where $[x, y] = xy - yx$. In the above mentioned paper by K.I. Beidar and A.V. Mikhalev [2] it was shown that if R_1 and R_2 satisfy the identity $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = 0$, then R is a PI-ring. In [6] and [7] M. Kepczyk and E.R. Puczyłowski established that if R_1 satisfies the identity $x^n = 0$ and R_2 is a PI-ring, then R is also a PI-ring.

It is also known that the question has a positive answer when some additional conditions are imposed on products of elements of R_1 and R_2 . Namely, it was shown by L.H. Rowen [11] in 1976 that one of such conditions is the following one: R_1 and

R_2 are both left (or right) ideals of R . Then M. Kepczyk and E.R. Puczyłowski [8] established that it is enough to assume that R_1 is a left or right ideal. Moreover, B. Felzenszwalb, A. Giambruno and G. Leal [3] extended this result even further by proving that the statement of the question holds in case $(R_1 R_2)^k \subset R_1$ or $(R_1 R_2)^k \subset R_2$ for some $k > 0$. They also imposed the following upper bound on the degree D' of the polynomial identity which is valid in R :

$$D' \leq a^a + 1 \text{ for } a = 8e(kd(d-1) - 1)^2(d-1)^2,$$

where R_1 and R_2 satisfy identities of degree d , and e is in the basis of the natural logarithms.

An \mathbb{F} -algebra B is said to *almost* satisfy some property if there exists a two-sided ideal I of B of finite codimension which satisfies this property. In 2008 M. Kepczyk [10] established that if R_1 and R_2 are almost nilpotent \mathbb{F} -algebras, then R is also a nilpotent \mathbb{F} -algebra. In a recent preprint [10] M. Kepczyk extended the mentioned result to the case of arbitrary identities f_1, f_2 with the following property: the conjecture by Beidar and Mikhalev holds for all rings R_1, R_2 satisfying f_1 and f_2 , respectively.

In Corollary 2.2 we improve the known upper bounds (see [1]) on the degree of identity of R over a field of positive characteristic. In Corollary 2.3 we show that the results of [1] hold in case R_1 and R_2 almost satisfy the following property: there are k and $i = 1, 2$ such that $(R_1 R_2)^k \subset R_i$. The main result of this paper unites Corollary 2.2 and 2.3 (see Theorem 2.1).

2. SYMMETRIC IDENTITIES

In 1993 A. Kemer [5] established that I.B. Volichenko's conjecture holds, which claims that any PI-algebra over a field of positive characteristic satisfies the symmetric identity

$$s_d = \sum_{\sigma \in S_d} x_{\sigma(1)} \cdots x_{\sigma(d)}$$

for some d , where S_d stands for the group of permutations.

We assume that $A = A_1 + A_2$, where A_1 and A_2 are PI-algebras over a field \mathbb{F} of positive characteristic p .

Theorem 2.1. *Let A_1 and A_2 satisfy the symmetric identities of degrees d_1 and d_2 , respectively. Assume that there exist $k > 0$ and vector spaces $V_i \subset A_i$ of finite codimensions t_i ($i = 1, 2$) such that $(V_1 V_2)^n$ is a subset of A_1 or A_2 . Then A satisfies the symmetric identity of degree*

$$D = ((d_1 + d_2 - 2)(kd_i + 2) - 1)((t_1 + t_2)(p - 1) + 1) + 1.$$

Corollary 2.2. *Let A_1 and A_2 satisfy the symmetric identities of degree d_1 and d_2 , respectively, and $(A_1 A_2)^k$ is a subset of A_i for $i = 1, 2$. Then A satisfies the symmetric identity of degree $D = (d_1 + d_2 - 2)(kd_i + 2)$.*

Corollary 2.3. *Let A_1 and A_2 satisfy the symmetric identities of degrees d_1 and d_2 , respectively. Assume A_1 and A_2 almost satisfy the following property: $(A_1 A_2)^k$ is a subset of A_1 or A_2 for some $k > 0$. Then A is a PI-algebra.*

Let us prove the theorem. Consider $s_D(a_1, \dots, a_D)$ for a_1, \dots, a_D from A . At first assume that $(A_1 A_2)^k \subset A_1$. Since s_D is linear and $A = A_1 + A_2$, without loss of generality we can assume that a_1, \dots, a_D are elements of $A_1 \cup A_2$.

Given $\sigma \in S_D$ and $i, j \geq 0$ with $i + j \leq D$, denote by $b_\sigma(i, j)$ the product $b_{\sigma(i+1)} \cdots b_{\sigma(i+j)}$ in case $j > 0$, and $b_\sigma(i, 0) = 1$. Similarly, we denote by $c_\sigma(i, j)$ the product $c_{\sigma(i+1)} \cdots c_{\sigma(i+j)}$ in case $j > 0$, and $c_\sigma(i, 0) = 1$. Denote by $\Delta_{r,s}$ the set of all sequences $(\beta_1, \gamma_1, \dots, \beta_m, \gamma_m)$ such that $m \geq 1$, $\beta_1, \gamma_m \geq 0$, $\gamma_1, \beta_2, \gamma_2, \dots, \beta_m \geq 1$ and $\beta_1 + \cdots + \beta_m = r$, $\gamma_1 + \cdots + \gamma_m = s$.

We start with the case of $t_1 = t_2 = 0$, i.e. $(A_1 A_2)^k$ is a subset of A_1 or A_2 . Therefore, $D = (d_1 + d_2 - 2)(kd_1 + 2)$. Without loss of generality we can assume that $a_1 = b_1, \dots, a_r = b_r$ are elements of A_1 and $a_{r+1} = c_1, \dots, a_D = c_s$ are elements of A_2 for some $r, s \geq 0$ satisfying the equality $r + s = D$. Thus

$$s_D(b_1, \dots, b_r, c_1, \dots, c_s) = \sum_{\delta \in \Delta_{r,s}} s_\delta,$$

where for $\delta = (\beta_1, \gamma_1, \dots, \beta_m, \gamma_m)$ we write $s_\delta = s_\delta(b_1, \dots, b_r, c_1, \dots, c_s)$ for

$$\sum_{\sigma \in S_r, \tau \in S_s} b_\sigma(0, \beta_1) c_\tau(0, \gamma_1) \cdots b_\sigma(\beta_1 + \cdots + \beta_{m-1}, \beta_m) c_\tau(\gamma_1 + \cdots + \gamma_{m-1}, \gamma_m).$$

If $\beta_i \geq d_1$ or $\gamma_i \geq d_2$ for some i , then $s_\delta = 0$ in A , since A_1 and A_2 satisfy the symmetric identities of degrees d_1 and d_2 , respectively.

Assume that $\beta_i \leq d_1 - 1$ and $\gamma_i \leq d_2 - 1$ for all i . For short, denote $b_\sigma(1) = b_\sigma(0, \beta_1), \dots, b_\sigma(m) = b_\sigma(\beta_1 + \cdots + \beta_{m-1}, \beta_m)$ and $c_\tau(1) = c_\tau(0, \gamma_1), \dots, c_\tau(m) = c_\tau(\gamma_1 + \cdots + \gamma_{m-1}, \gamma_m)$. Obviously, $b_\sigma(1), \dots, b_\sigma(m)$ are elements of $A_1 \cup \{1\}$ and $c_\tau(1), \dots, c_\tau(m)$ are elements of $A_2 \cup \{1\}$. Using new notations we obtain

$$s_\delta = \sum_{\sigma \in S_r, \tau \in S_s} b_\sigma(1) c_\tau(1) \cdots b_\sigma(m) c_\tau(m).$$

Let us recall that $\beta_1 + \cdots + \beta_m + \gamma_1 + \cdots + \gamma_m = D$. Thus $D \leq m(d_1 + d_2 - 2)$ and $m \geq kd_1 + 2$.

For fixed δ , k and some $i \geq k$, $\sigma \in S_r$, $\tau \in S_s$, we write $w_{\sigma,\tau}(i)$ for the product

$$b_\sigma(i - k + 1) c_\tau(i - k + 1) \cdots b_\sigma(i) c_\tau(i).$$

By the condition of the theorem, $w_{\sigma,\tau}(i) \in A_1$ in case $k < i < m$. Since $m \geq kd_1 + 2$, we can rewrite s_δ as follows:

$$s_\delta = \sum b_\sigma(1) c_\tau(1) \left(\sum w_{\sigma,\tau}(k+1) w_{\sigma,\tau}(2k+1) \cdots w_{\sigma,\tau}(kd_1+1) \right) \\ * b_\sigma(kd_1+2) c_\tau(kd_1+2) \cdots b_\sigma(m) c_\tau(m),$$

where the first sum ranges over all $\sigma \in S_r$, $\tau \in S_s$ with $\sigma(\beta_1 + \cdots + \beta_{k+1}) < \cdots < \sigma(\beta_1 + \cdots + \beta_{kd_1+1})$ and the second sum interchanges w -as. Therefore, the second sum is zero in A and s_δ is zero as well. Thus A satisfies the identity $s_D(a_1, \dots, a_D)$.

Consider the general case of $t_1 \geq 0$ and $t_2 \geq 0$. There exists a vector space U_i of dimension t_i such that A_i is the direct sum of V_i and U_i ($i = 1, 2$). Let v_{i1}, v_{i2}, \dots be an \mathbb{F} -basis for V_i and u_{i1}, \dots, u_{it_i} be an \mathbb{F} -basis for U_i . Without loss of generality we can assume that a_1, \dots, a_D are elements of the bases under consideration. Denote by l_i the number of elements of the set a_1, \dots, a_D from the basis of U_i . If $l_i \geq pt_i - t_i + 1$, then $s_D(a_1, \dots, a_D) = 0$. This remark follows from the fact that if w_1, \dots, w_{l_i} are some elements from the bases of U_i , where l_i is as above, then $s_D(w_1, \dots, w_{l_i}) = p!h = 0$ for some $h \in A_i$.

Assume that $l_i \leq pt_i - t_i$ for $i = 1, 2$. Then $s_D(a_1, \dots, a_D)$ is the sum of all elements of the following form:

$$\sum y_0 z_1 y_1 \cdots z_q y_q,$$

where z_1, \dots, z_q are monomials of elements of the bases of U_1 and U_2 and y_0, \dots, y_q are monomials of elements of the bases of V_1 , and the sum interchanges all letters of monomials y_0, \dots, y_q . Here only monomials y_1 and y_q can be empty. Since degree of $z_1 \dots z_q$ is $l_1 + l_2 \leq (t_1 + t_2)(p - 1)$, there is j such that the degree of y_j is greater or equal to $(d_1 d_2 - 2)(k d_1 + 2)$. The case of $t_1 = t_2 = 0$ implies that $s_D(a_1, \dots, s_D) = 0$.

The proof in the case of $(A_1 A_2)^k \subset A_2$ is the same.

ACKNOWLEDGEMENTS

The second author was supported by RFFI 14-01-00068 and FAPESP No. 2013/15539-2. He is grateful for this support.

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